❖ *Proofs and Deductions* ❖

3.14. Conditional Deduction

1. Conditional Deduction: Hypothetical Reasoning Revisited. Indirect deduction introduced us to hypothetical reasoning: deducing what *would* happen, if a certain sentence *were* accepted as true. A similar approach underlies a method of deduction devoted solely to conditionals.

Here, instead of hypothetically assuming the negation of the conclusion (as in ID), we assume the **antecedent** of the conditional sentence we wish to deduce – exploring where that antecedent would lead. If we succeed in tracing a deductive trail from the assumed antecedent to the **consequent** of the conditional, we have shown that **if the antecedent is true, then the consequent is true** – such hypothetical reasoning thereby establishing that the conditional is true.

For obvious reasons we call this **Conditional Deduction** ("**CD**" for short).

The following is an intuitively valid argument with a conditional conclusion.

> 1. We're having either ice cream or cake.
>
> ---
>
> (So,) If we're not having ice cream, then we're having cake.

Conditional Deduction provides a natural way of deducing this conditional conclusion from the premise.

We first restate the argument in formal language.

> **P**: We're having ice cream
> **Q**: We're having cake
>
> 1. $(P \vee Q)$
>
> ---
>
> $\therefore\ (\sim P \rightarrow Q)$

The conditional deduction (CD) of the argument is set up like so.

1.	$(P \vee Q)$	
		Get $(\sim P \rightarrow Q)$ (CD)

Then the hypothetical reasoning begins – as with ID, marked as hypothetical by occurring within a box. First the **Assumption of the Conditional Deduction** (ACD): suppose we *did* have the antecedent.

1.	$(P \vee Q)$	
		Get $(\sim P \rightarrow Q)$ (CD)
2.	$\sim P$	ACD

From lines (1) and (2), “Q” follows by simple $\vee$–.

1.	$(P \vee Q)$	
		Get $(\sim P \rightarrow Q)$ (CD)
2.	$\sim P$	ACD
3.	Q	1, 2, $\vee$–

But having demonstrated that if “~P” then “Q”, we’ve established the truth of the whole conditional. The hypothetical reasoning is then complete.

1.	$(P \vee Q)$	
		Get $(\sim P \rightarrow Q)$ (CD)
2.	$\sim P$	ACD
3.	Q	1, 2, $\vee$–
4.	$(\sim P \rightarrow Q)$	2, 3, CD

The conclusion of a Conditional Deduction is justified by citing the **assumption of the antecedent** (here, line 2), and the **consequent** deduced from that assumption (here, line 3).

Beyond their appeal to hypothetical reasoning, ID and CD are alike in other respects as well. First: as with IDs, once a CD box is closed all lines in that box become **unusable**. No rule can be applied later to a line in a closed box.

And like IDs, CDs can be used **recursively**: in the midst of one CD we can start another, embedding CDs within CDs. The following is a simple example.

1. We'll have either ice cream, or cake, or pie.

∴ If we don't have ice cream, then if we don't have cake we'll have pie.

1. $(P \vee (Q \vee R))$

∴ $(\sim P \rightarrow (\sim Q \rightarrow R))$

1.	$((P \vee (Q \vee R))$	
		~~Get~~: $(\sim P \rightarrow (\sim Q \rightarrow R))$ (CD)
2.	$\sim P$	ACD
3.	$(Q \vee R)$	1, 2, $\vee-$
		~~Get~~: $(\sim Q \rightarrow R)$ (CD)
4.	$\sim Q$	ACD
5.	R	3, 4, $\vee-$
6.	$(\sim Q \rightarrow R)$	4, 5, CD
7.	$(\sim P \rightarrow (\sim Q \rightarrow R))$	2, 6, ID

Conditional deduction is fundamentally unlike indirect deduction in one way, however: whereas ID is suitable for deducing any type of sentence, CD is only useful for deducing a conditional.

That point marks a change in our deductive strategy. While the advent of ID led to a default strategy of automatically reaching for ID, **for any conditional conclusion we now automatically use CD** (unless an easier way of getting that conclusion is obvious).

2. More Strategy: Conditional and Indirect Deductions. Beyond that default strategy, we note here some further, less general tactics for deducing a conditional.

Recall that the conditional "(P → Q)" is semantically equivalent to the disjunction "(~P ∨ Q)".

P	Q	~P	**(~P ∨ Q)**	**(P → Q)**
1	1	0	**1**	**1**
1	0	0	**0**	**0**
0	1	1	**1**	**1**
0	0	1	**1**	**1**

So we should expect "(P → Q)" to follow from the same sentence(s) that "(~P ∨ Q)" does. But "(~P ∨ Q)" follows from "~P", and likewise from "Q", by simple ∨+.

$$\frac{\sim P}{(\sim P \vee Q)} \qquad\qquad \frac{Q}{(\sim P \vee Q)}$$

And indeed "(P → Q)" is likewise deducible from each of these premises.

1. ~P

Get: (P → Q) (CD)

2.	P	ACD
3.	(P ∨ Q)	2, ∨+
4.	Q	1, 4, ∨–
5.	(P → Q)	(2, 4, CD)

1.	Q	
		Get: $(P \rightarrow Q)$ (CD)
2.	P	ACD
3.	Q	1, R
4.	$(P \rightarrow Q)$	(2, 3, CD)

So when a certain conditional is needed in a deduction, it's tactically handy to keep in mind that the conditional deducible from either its consequent or the negation of its antecedent.

> ***Deduction Strategy***: A conditional is deducible from its **consequent**, as well as from the **negation of its antecedent**.

The arrival of conditionals also brings a new wrinkle to our **indirect deduction** strategy. A time-saving measure for IDs was to use a sentence we already have as half of the needed contradiction, and derive the other sentence – cutting our deductive work in half.

But if one sentence we already have is the **negation of a conditional**, our savings in labor is doubled. For then it remains only to secure the other half of the contradiction – a conditional. Since conditional deduction brings the antecedent of that conditional (as the ACD), to complete the contradiction all that remains is deducing the other half of the conditional – its consequent.

> ***Deduction Strategy***: In an Indirect Deduction, if the **negation of a conditional** is available, **use** that **as half of the contradiction**, and deduce the other half (the conditional) using CD.[1]

As illustrations of this point, note that the negation of a conditional is logically equivalent to a **conjunction**: of the **antecedent**, and the **negation of the consequent**. For example, "~(P → Q)" is equivalent to "(P ∧ ~Q)".

P	Q	~Q	(P → Q)	**~(P → Q)**	**(P ∧ ~Q)**
1	1	0	1	**0**	**0**
1	0	1	0	**1**	**1**
0	1	0	1	**0**	**0**
0	0	1	1	**0**	**0**

But from "(P ∧ ~Q)" both its left and right parts, "P" and "~Q," follow immediately by ∧–. Likewise, from the equivalent "~(P → Q)" both "P" and "~Q" follow – as deductions show.

1.	~(P → Q)	
		~~Get:~~ P (ID)
2.	~P	AID
		~~Get~~ (P → Q) (CD)
3.	P	ACD
4.	(P ∨ Q)	3, ∨+
5.	Q	4, 6, ∨–
6.	(P → Q)	3, 5, CD
7.	~(P → Q)	1, R
8.	P	2, 6, 7, ID

[1] Following a suggestion from Kalish and Montague 1964: 26 (#5).

1. ~(P $\rightarrow$ Q)

		Get: ~Q (ID)
2.	~~Q	AID
3.	Q	
		Get (P $\rightarrow$ Q) (CD)
3.	P	ACD
4.	Q	3, R
5.	(P $\rightarrow$ Q)	3, 4, CD
6.	~(P $\rightarrow$ Q)	1, R
7.	~Q	2, 5, 6, ID

In general: whenever we have the **negation of a conditional** it's tactically useful to keep in mind that both the **antecedent** and **negation of the consequent** are deducible from that sentence.

Deduction Strategy: from the **negation of a conditional** both the antecedent of that conditional and the negation of its consequent are deducible.

Summary: Conditional Deduction (CD)

- Write (**CD**) next to the “Get” line, as a reminder.
- Immediately following the “Get” line, begin a **box**, in which the Conditional Deduction occurs.
- The first line in the CD box is the **Assumption of the Conditional Deduction** (**ACD**): the **antecedent** of the sentence on the “Get” line.
- Using deductive rules on all available lines (premises and ACD), deduce the **consequent** of the sentence on the “Get” line.
- Once the consequent has been deduced, **close** the CD box. (When the CD box is closed, no rules can be applied to any line in that box. These sentences become “**unusable**”.)
- Beneath the CD box write the conclusion of the argument (the sentence on the “Get” line). The justification for this conclusion cites two lines: the **ACD**, and the **consequent**. These two numbers are followed by “**CD**”.

Deduction Strategy:

- If the conclusion of an argument is a **conditional**, automatically use **CD** for that argument. If the conclusion of the argument is any other type of sentence (sentence letter, negation, conjunction, or disjunction), use **ID**.
- **A conditional follows validly from** (i) the **negation of** its **antecedent**, and (ii) **from** its **consequent**.
- **The negation of a conditional entails** both (i) its **antecedent**, and (ii) the **negation of its consequent**.
- In an **Indirect Deduction** where the **negation of a conditional** is available, **use** that negation **as half of the** needed **contradiction**, and derive the other half (the conditional) using CD.